Magneto-optics in bi-gyrotropic garnet waveguide

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In many works, the magneto-optical effects for optical frequencies are normally described by using an electric dipole approximation. Nevertheless, this approximation is problematic in far infrared region and for ultrathin films in the optical band. For this reason, dispersion relations have been derived for guided modes in a monolayer system at transverse geometry with bi-gyrotropic anisotropy on the base of matrix elements. The permeability tensor effect on waveguiding is analysed in detail. Experimental possibilities of the dark mode spectroscopy for permeability tensor study in mentioned structures are briefly discussed.

Keywords: magneto-optics, guided modes, bi-gyrotropic anisotropy.

1. Introduction

The electromagnetic wave interaction in magnetic anisotropic layered structures is rigorously described by Yeh's 4×4 matrix algebra [1]. Magnetic anisotropy is completely specified by the relative permeability and the relative permittivity tensors. The boundary conditions for electric and magnetic fields at the layer interfaces are advantageously written in the 4×4 matrix form, which results in the equation

\[
\begin{pmatrix}
A_1^{(0)} \\
A_2^{(0)} \\
A_3^{(0)} \\
A_4^{(0)}
\end{pmatrix} =
\begin{pmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\begin{pmatrix}
A_1^{(s)} \\
A_2^{(s)} \\
A_3^{(s)} \\
A_4^{(s)}
\end{pmatrix},
\]  

where \( M \) is, so-called, total matrix of the layered system.

The reflection and transmission coefficients can be defined and expressed in terms of the \( M \)-matrix elements. A guided mode in the layered system corresponds to the special boundary conditions for complex amplitude of \( j \)-th eigenmode, \( A_1^{(0)} = A_2^{(s)} = 0 \) and \( A_3^{(0)} = A_4^{(s)} = 0 \), i.e., no incident waves exist in sub- or superstrate. Equation (1) leads to the general condition (dispersion law) for guided modes,

\[
M_{11}M_{33} - M_{13}M_{31} = 0,
\]

where \( M_{ij} \) are the elements of Yeh's total matrix \( M \).

2. Eigenmodes in bi-gyrotropic layer

Taking into account the Maxwell's equations, the generalised first-order linear differential system for bi-gyrotropic media can be advantageously written in compact 4×4 matrix form [2]

\[
\frac{\partial}{\partial z} f' = -i \frac{\omega}{c} C f',
\]

where \( \omega \) is the angular frequency, \( c \) is the light velocity in vacuum, and \( f' = (E_{0x}, H_{0y}, E_{0y}, H_{0x})^T \) is a column vector that contains only tangential components of normalised electric and magnetic complex amplitude, respectively. The matrix \( C \) is the coupling matrix with elements (note that the plane of incidence is considered as \( yz \) plane – see Fig. 1) of the form

![Fig. 1. Schematic diagram of waveguide structure and co-ordinate system.](image)

\( \epsilon^{(0)} \) Superstrate, \( \epsilon^{(1)} \) MO layer, \( \epsilon^{(2)} \) Substrate.
where $\beta = K$, is the tangential component of complex propagation constant and $k_v$ is the wavenumber in vacuum. If the matrix $C$ is approximately independent of $z$ (transverse co-ordinate) over some short interval of $\xi$, then there are four periodic solutions of Eq. (3) of the form

$$f_{j}(\xi) = v_{j}(0)e^{ik_v\xi}. \quad (5)$$

The four eigenvalues $K_{ij}$ and corresponding eigenvectors $v_{j}$ of matrix $C$ can be found by many standard procedures [3]. Arbitrary wave propagating in an anisotropic medium can be considered as a superposition of eigenmode waves.

The simplest situation occurs at the transversal configuration. In this case, the relative permittivity $\varepsilon$ and relative permeability $\mu$ are defined as

$$\varepsilon = \begin{pmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & -i\varepsilon_1 \tau \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_0 & -i\mu_1 \tau \\ 0 & 0 & \mu_0 \end{pmatrix} \quad (6, 7)$$

where $Q_{\tau}$ is the Voigt parameter (linear magneto-optical constant). The coupling matrix $C$ now takes the following form

$$C = \begin{pmatrix} \frac{\mu_2}{\mu_3} \frac{\beta}{k_v} & \mu_{22} - \frac{\mu_{23} \beta}{\mu_3} \\ \varepsilon_{11} - \frac{1}{\mu_3} \left( \frac{\beta}{k_v} \right)^2 & \mu_{11} - \frac{1}{\mu_3} \left( \frac{\beta}{k_v} \right)^2 \\ 0 & 0 \\ 0 & \varepsilon_{22} - \frac{1}{\mu_3} \left( \frac{\beta}{k_v} \right)^2 \end{pmatrix} \quad (8)$$

are mutually independent – they satisfy separately Maxwell’s equations and no TE–TM coupling occurs.

It is useful to transform the set of differential equations (3) into a diagonal form. This can be done, if we found all eigenvalues of coupling matrix $C$. They can be determined from the characteristic equation, which is derived by substituting Eq. (5) into Eq. (3)

$$\det(C - \kappa I) = 0. \quad (9)$$

The $\kappa = (K_{11}, K_{23}, K_{33}, K_{44})^T$ is a vector containing the eigenvalues and $I$ is a 4×4 identity matrix. Expanding the determinant we get the quartic polynomial equation in $K_\alpha$ and the four eigenvalues may be obtained analytically as

$$K_{11} = -K_{22} = k_v\sqrt{\varepsilon_0 \mu_0 (\alpha^2 - P_T^2)}, \quad K_{33} = -K_{44} = k_v\sqrt{\varepsilon_0 \mu_0 (\alpha^2 - Q_T^2)}, \quad P_T = \frac{\mu_{11} \mu_{23} - \mu_{21} \mu_{13}}{\mu_0}, \quad Q_T = \frac{\mu_{21} \mu_{33} - \mu_{23} \mu_{13}}{\mu_0}, \quad (10)$$

Eigenvectors $v_{j}$ can be found for each eigenvalue $K_{ij}$ by solving three of the four simultaneous equations represented by the matrix equation

$$\frac{\omega}{c} C v_{j} = K_{ij} v_{j}, \quad (11)$$

Now, the transformation matrix $T = (x_1, x_2, x_3, x_4)$, which includes only tangential components of eigenpolarisation vectors [see the structure of Eq. (3) and Eq. (12)], can be specified then. Using this matrix we can transform $f^i$ to $g$ by the relation $f^i = Tg$, so that the Eq. (3) is transformed.


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into the form $g(z) = \exp(i \kappa z)$. From the point of view of mathematical theory, it is so called similarity reduction; in physics it means, that we changed the representation of the field from field components to eigenmode amplitudes. To illustrate some results, we present a complete analytical form of eigenpolarisation vectors for electric field [4]

$$
\begin{align*}
\beta_{\text{TE}} &= (1,0,0)^T, \\
\beta_{\text{TM}} &= \begin{cases} 
0 & iQ_T - \sqrt{(1-\alpha^2)(\alpha^2-Q_T^2)} \\
\alpha^2 & iQ_T + \sqrt{(1-\alpha^2)(\alpha^2-Q_T^2)} 
\end{cases} 
\end{align*}
$$

(12)

The linear TE polarisation corresponds to $K_{x1,2}$. For TM-polarised waves the electric field vectors trace the ellipses in the $yz$ plane.

3. Bi-gyrotrropic thin film waveguide at transverse geometry

Transverse magneto-optical (MO) geometry (the magnetisation is perpendicular to the plane of incidence) is the most suitable for bi-gyrotrropy study, because in this case the electric and magnetic gyrotrropy effects are separable in linear approximation [5]. The total matrix $M$ of the layer system is block-diagonised. First $2\times2$ block corresponds to the TE polarisation and the second one to the TM polarisation, so that we have two mutually independent dispersion relations for TE and TM polarised guided modes (see Eq. (2)),

$$
M_{11}(\beta_m) = 0, \quad M_{33}(\beta_m) = 0,
$$

(13)

or equivalently expressed through the reflection coefficients [6],

$$
\begin{align*}
F_{\text{TE}}(\beta_m) &= 1 - r_{21}^{(10)}r_{22}^{(12)}\exp(-i2K_{z1}^{(n)}d^{(n)}) = 0, \\
F_{\text{TM}}(\beta_m) &= 1 - r_{43}^{(10)}r_{44}^{(12)}\exp(-i2K_{z3}^{(n)}d^{(n)}) = 0
\end{align*}
$$

(14)

We noted that the description by elements of $M$ matrix is suitable for a computer simulation of waveguiding effects for complicated multilayered structures. On the contrary, dispersion Eq. (14) allows detailed description of the guided radiation from physical point of view.

The normalised longitudinal propagation constant (effective refractive index),

$$
\frac{c}{\omega}b = \frac{c}{\omega}(\beta' - i\beta'') \equiv N_{\text{eff}} = n_{\text{eff}} - ik_{\text{eff}},
$$

(15)
can be determined by numerical calculation as the complex root of Eqs. (13) and (14). Only in the case of lossless materials, $\beta$ becomes a pure real quantity related to the usual notation, $\beta = \beta' = (\omega/c)n_{\text{eff}}$. The imaginary part $\beta''$ describes the exponential decrease of the field amplitudes along the waveguide caused by absorption in the case of guided modes and/or by radiation as the superstrate or substrate leaky modes.

As the example, we will consider the following planar structure consisting of the one bi-gyrotrropic MO thin film ((BiGe)$_3$(Fe)$_{12}$O$_{30}$, $\varepsilon_1^{(1)} = 4.8500 - i4.4360\times10^{-2}$, $\varepsilon_2^{(1)} = 2\times10^{-4}$, $\mu_1^{(1)} = 102$, $\mu_2^{(1)} = 1\times10^{-2}$) with GGG garnet as a substrate ($\varepsilon_1^{(2)} = 19.6493^2$ at wavelength $\lambda = 632.8$ nm [7]). Superstrate is formed by air.

Our numerical analysis is based on the optimisation of the objective function $A_{\text{WG}} = \text{abs}(M_{11})$, $i = 1, 3$ for TE or TM polarisation, respectively ($A_{\text{WG}}$ - Amplitude of Waveguide Term). Thus the minimum of $A_{\text{WG}}$ corresponds to the waveguiding resonant state (waveguiding mode). We use at first constrained minimisation method, which has been focused on the solution of the Kuhn-Tucker (KT) equations. The KT equations are necessary conditions for optimality for constrained optimisation problems. The solution of the KT equations forms the basis to many non-linear programming algorithms. We use the Sequential Quadratic Programming (SQP) method which represents state-of-the-art in nonlinear programming methods. An overview of SQP is found in Refs. 8 and 9. Starting estimates of parameters are obtained with the help of perturbation method - there is neglected an imaginary part of $\beta$ (non-important contribution), although the index of refraction is complex. The first step (constrained problem) is very useful for a final minimisation, because waveguide equation can be generally polynomial function, and the knowledge of the finding area enables to avoid jumps into the other minima (resonant states). Furthermore boundary constraints applied to unknown parameters are very helpful to decreasing computational time (number of iteration steps). The final minimisation is performed by simplex search algorithm [3] and starting estimates are adopted from constrained minimisation process.

Complete results of numerical analysis of TE polarisation (influence diagonal elements of permeability tensor together with off-diagonal ones of $\mu$ proves only at TE polarisation [10]), as we can see in Tables 1 and 2. We solved four types of waveguiding structures - isotropic nonabsorbing (INA), isotropic absorbing (IA), gyroelectric (GE), and bi-gyrotrropic (BG) waveguide systems, respectively.

Table 1 shows that real parts of effective indices in absorbing structure are smaller than for lossless medium. This is due to the fact, that the angle of ray course in absorbing medium (measured in the normal to the incidence plane) is smaller then in lossless medium. This shift is about $2\times10^{-5}$ and the effect is more significant for modes of higher order. Imaginary parts of effective indices should be followed and corresponded to their real parts (Table 2). In nonabsorbing structure, the values of $k_{\text{eff}}^m$ elements are of the order of $10^{-19}$, what is an inaccuracy of numerical computation. In IA case one can see the important result: modes of higher
Table 1. Numerical analysis of bi-gyrotropic waveguide structure – real parts of complex effective indices.

<table>
<thead>
<tr>
<th>m</th>
<th>INA $n_{\text{eff}}$</th>
<th>IA $n_{\text{eff}}$</th>
<th>GE $n_{\text{eff}}$</th>
<th>BG $n_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.185169</td>
<td>2.185167</td>
<td>2.185167</td>
<td>2.207006</td>
</tr>
<tr>
<td>1</td>
<td>2.133572</td>
<td>2.133562</td>
<td>2.133562</td>
<td>2.15137</td>
</tr>
<tr>
<td>2</td>
<td>2.047774</td>
<td>2.047730</td>
<td>2.047730</td>
<td>2.068535</td>
</tr>
</tbody>
</table>

INA: $\varepsilon_0^{(1)} = 4.8501 - i\alpha_0$, $\varepsilon_0^{(1)} = 0$, $\mu_0^{(1)} = 10$, $\mu_0^{(1)} = 0$, $N^{(1)} = 2.2023 - i\alpha_0$

IA: $\varepsilon_0^{(1)} = 4.8500 - i4.4360 \times 10^{-2}$, $\varepsilon_0^{(1)} = 0$, $\mu_0^{(1)} = 10$, $\mu_0^{(1)} = 0$, $N^{(1)} = 2.2023 - i0.071 \times 10^{-2}$

GE: $\varepsilon_0^{(1)} = 4.8500 - i4.4360 \times 10^{-2}$, $\varepsilon_1^{(1)} = 2 \times 10^{-3}$, $\mu_0^{(1)} = 10$, $\mu_1^{(1)} = 0$, $N^{(1)} = 2.2023 - i0.071 \times 10^{-2}$

BG: $\varepsilon_0^{(1)} = 4.8500 - i4.4360 \times 10^{-2}$, $\varepsilon_1^{(1)} = 2 \times 10^{-3}$, $\mu_0^{(1)} = 102$, $\mu_1^{(1)} = 1 \times 10^{-5}$, $N^{(1)} = 2.242 - i0.0172 \times 10^{-2}$

Table 2. Numerical analysis of bi-gyrotropic waveguide structure – imaginary parts of complex effective indices.

<table>
<thead>
<tr>
<th>m</th>
<th>INA $k_{\text{eff}}$</th>
<th>IA $k_{\text{eff}}$</th>
<th>GE $k_{\text{eff}}$</th>
<th>BG $k_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$9.968 \times 10^{-19}$</td>
<td>$1.0071 \times 10^{-2}$</td>
<td>$1.0071 \times 10^{-2}$</td>
<td>$1.0181 \times 10^{-2}$</td>
</tr>
<tr>
<td>1</td>
<td>$3.485 \times 10^{-19}$</td>
<td>$1.0037 \times 10^{-2}$</td>
<td>$1.0037 \times 10^{-2}$</td>
<td>$1.0179 \times 10^{-2}$</td>
</tr>
<tr>
<td>2</td>
<td>$2.538 \times 10^{-19}$</td>
<td>$9.729 \times 10^{-3}$</td>
<td>$9.729 \times 10^{-3}$</td>
<td>$9.986 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

(INA, IA, GE, BG – the same parameters as for Table 1).

order are attenuated less than modes of lower orders. It is potential that optical path across the absorbing layer is more considerable for higher mode orders than for lower mode orders due to the penetrations into the sub- and superstrate and the following Goos-Hänchen shifts. These results are in good agreement with Refs. 11 and 12. The comparison of IA and GE cases gives good agreement with the theory, because off-diagonal elements of permittivity tensor should not affect to guided modes in TE polarization.

Completely different situation arises in bi-gyrotropic case. Brew up a redistribution of refractive index through the permeability tensor. Both the real and also imaginary parts of effective indices are shifted to the higher values (imaginary part even above a volume extinction coefficient of absorbing layer), because the optical density increased. One can say, that the effective indices increase with influence of permeability tensor. For instance, by parameters $m = 0$, $d^{(1)} = 1$ µm, the shift of diagonal elements $\mu_0^{(1)}$ in the frame of 2% and the shift of off-diagonal elements $\mu_0^{(1)}$ of $1 \times 10^{-5}$, respectively, the corresponding change of effective

![Fig. 2. Dependence of the first estimation of amplitude waveguide term (AWgT_e) on real part of effective index.](image)

![Fig. 3. (a) Detail of the first estimation of resonant waveguide states for n_{eff}. (b) Detail of the first estimation of resonant waveguide states for k_{eff}.](image)
index will be $2.1839 \times 10^{-2}$ for $n_{\text{eff}}$ and $1.1 \times 10^{-4}$ for $k_{\text{eff}}$. This fact is very important by application of dark mode spectroscopy (DMS) method, because these values already very affect the determination of the thickness and refractive index of thin layers (standard accuracy of DMS is about $10^{-4}$ to $10^{-5}$ [13]).

The optimisation process – finding of waveguiding resonant states – for all cases was finished by limit of objective function in order about $10^{-15}$ and waveguide equation (12) takes in the resonant states values $10^{-15}$ for real part and $10^{-16}$ for imaginary part of this equation. These values offer good check for correct solution of the considered problem. The graphical outputs are showed for better illustration of these computations in the following figures. Figures 2 and 3 show the first estimated course of waveguide amplitude term (neglecting of $k_{\text{eff}}$).

In Fig. 2, one can see typical resonant states of guided modes in planar structure for $n_{\text{eff}}$. Figure 3(a) shows this situation more perfect – in the detail one can see that the curve does not touch to zero in minima. Waveguide equation is not fulfilled from point of view of mathematics, but physically it gives obvious conception about attenuation process. The analogous situation for $k_{\text{eff}}$ is described in Fig. 3(b). Figures 4(a) and 4(b) specify the courses of resonant states, which were determined from optimisation process. In comparison with Fig. 2 and Figs. 3(a) and 3(b), the minima at resonant states reach to zero for both real and imaginary parts of effective indices and that are why the waveguide condition is fulfilled.

4. Conclusions

At transverse geometry the propagation of the light in planar structures can be described for TE and TM waves as mutually independent. It gives us the new ideas for experimental set-up based on DMS. The shifts of resonant states of guided modes generated by diagonal elements of permittivity and permeability tensors can by easy specified by prism coupling technique. As regards the off-diagonal elements the direct observation of dark line changes is non-measurable by the DMS experiment.

Because of linear approximation the TE guided modes are characterised by linear polarisation and TM modes are elliptically polarised waves, one can analyse the off-diagonal element contribution by the measurement of wave polarisation states. On the base of external field modulation one can switch the position of a magnetization vector in the planar structure. As the result we observe the influence of linear elements of permittivity (permeability) tensor on rotation and ellipticity of guided modes.

The role of boundaries in absorbing waveguides and the damping as function of guided mode order can be easy explained by Goos-Hanchen shifts.

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References


